EE 508
Lecture 7

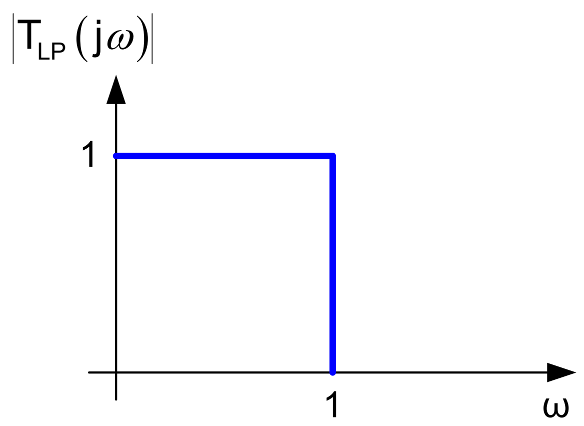
The Approximation Problem

Review from Last Time

The Approximation Problem

The goal in the approximation problem is simple, just want a function $T_A(s)$ or $H_A(z)$ that meets the filter requirements.

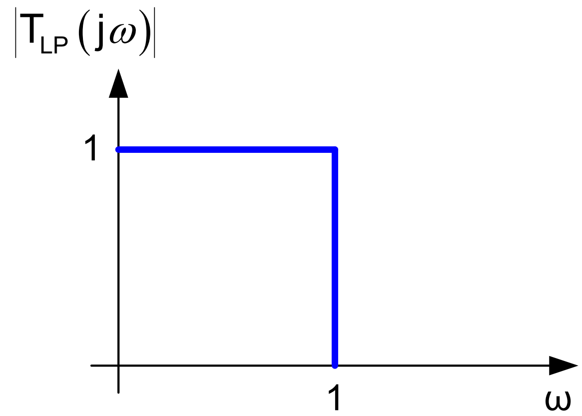
Will focus primarily on approximations of the standard normalized lowpass function



- Frequency scaling will be used to obtain other LP band edges
- Frequency transformations will be used to obtain HP, BP, and BR responses

Review from Last Time

The Approximation Problem



$$T_A(s) = ?$$

$T_A(s)$ is a rational fraction in s

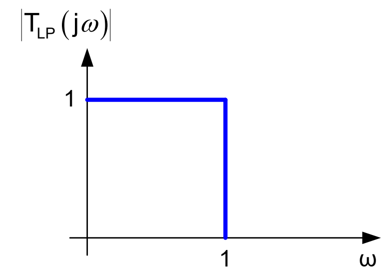
$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

Rational fractions in s have no discontinuities in either magnitude or phase response

No natural metrics for $T_A(s)$ that relate to magnitude and phase characteristics (difficult to meaningfully compare $T_{A1}(s)$ and $T_{A2}(s)$)

Review from Last Time

The Approximation Problem



Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares (Cost Function Minimizations)
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Thompson

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

$$T(j\omega) = \frac{\left[F_1(\omega^2)\right] + j\left[\omega F_2(\omega^2)\right]}{\left[F_3(\omega^2)\right] + j\left[\omega F_4(\omega^2)\right]}$$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left[F_1(\omega^2)\right]^2 + \omega^2\left[F_2(\omega^2)\right]^2}{\left[F_3(\omega^2)\right]^2 + \omega^2\left[F_4(\omega^2)\right]^2}}$$

Thus $\left|T(j\omega)\right|$ is an even function of ω

It follows that $\left|T(j\omega)\right|^2$ is a rational fraction in ω^2 with real coefficients

Since $\left|T(j\omega)\right|^2$ is a real variable, natural metrics exist for comparing approximating functions to $\left|T(j\omega)\right|^2$

Review from Last Time

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

If a desired magnitude response is given, it is common to find a rational fraction in ω^2 with real coefficients, denoted as $H_A(\omega^2)$, that approximates the desired magnitude squared response and then obtain a function $T_A(s)$ that satisfies the relationship $|T_A(j\omega)|^2 = H_A(\omega^2)$

$H_A(\omega^2)$ is real so natural metrics exist for obtaining $H_A(\omega^2)$

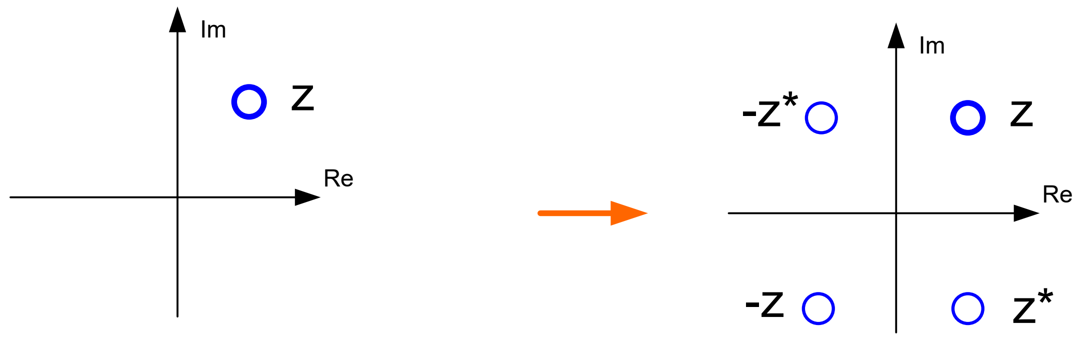
$$H_A(\omega^2) = \frac{\sum_{i=0}^{2l} c_i \omega^{2i}}{\sum_{i=0}^{2k} d_i \omega^{2i}}$$

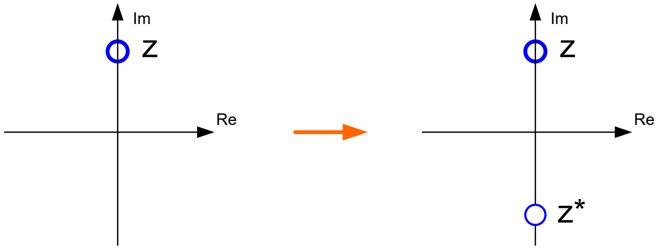
Obtaining $T_A(s)$ from $H_A(\omega^2)$ is termed the inverse mapping problem

But how is $T_{\underline{A}}(s)$ obtained from $H_A(\omega^2)$?

Review from Last Time

Observation: If z is a zero (pole) of $H_A(\omega^2)$, then $-z$, z^*, and $-z^*$ are also zeros (poles) of $H_A(\omega^2)$





Thus, roots come as quadruples if off of the axis and as pairs if they lay on the axis

Review from Last Time

Inverse MappingTheorem: If $H_A(\omega^2)$ is a rational fraction with real coefficients with no poles or zeros of odd multiplicity on the real axis, then there exists a real number H_0 such that the function

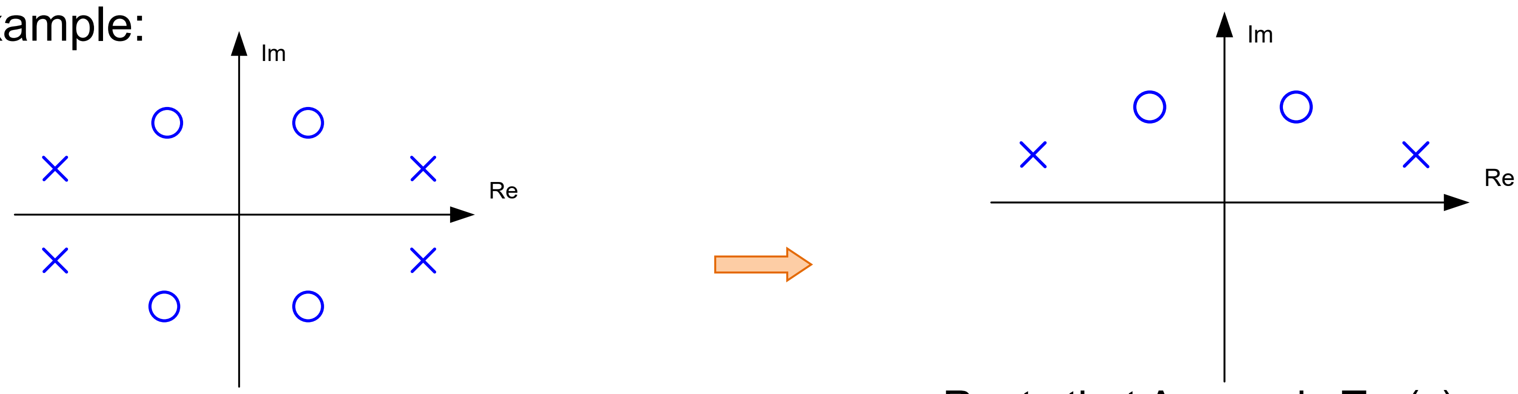
$$T_{AM}(s)=\frac{H_0(s\text{-}jz_1)(s\text{-}jz_2)\bullet...\bullet(s\text{-}jz_m)}{(s\text{-}jp_1)(s\text{-}jp_2)\bullet...\bullet(s\text{-}jp_n)}$$

is a minimum phase rational fraction with real coefficents that satisfies the relationship

$$\left|T_{AM}(j\omega)\right|=\sqrt{H_A(\omega^2)}$$

where $\{z_1, z_2, \ldots z_m\}$ are the upper half-plane zeros of $H_A(\omega^2)$ and exactly half of the real axis zeros,
and where where $\{p_1, p_2, \ldots p_n\}$ are the upper half-plane poles of $H_A(\omega^2)$ and exactly half of the real axis poles.

Example:



Roots of $H_A(\omega^2)$

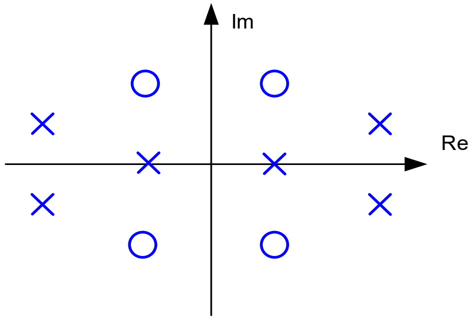
Roots that Appear in $T_{AM}(s)$

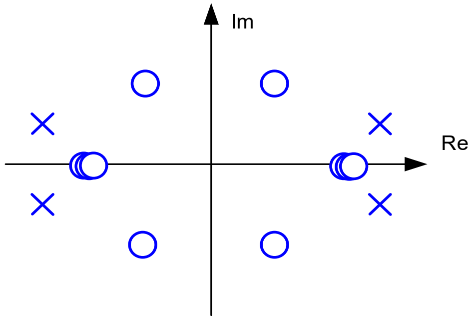
(but multiplied by j)

Theorem: If $H_A(\omega^2)$ is a rational fraction of order 2m/2n with real coefficients with one or more **poles** on the real axis that are of odd multiplicity, then there is no inverse mapping to a rational fraction T(s) with real coefficients that satisfies the relationship $\left|T(j\omega)\right| = \sqrt{H_A(\omega^2)}$

Theorem: If $H_A(\omega^2)$ is a rational fraction of order 2m/2n with real coefficients with one or more **zeros** on the real axis that are of odd multiplicity, then there is no inverse mapping to a rational fraction T(s) with real coefficients that satisfies the relationship $\left|T(j\omega)\right| = \sqrt{H_A(\omega^2)}$

Example where inverse mapping does not exist:





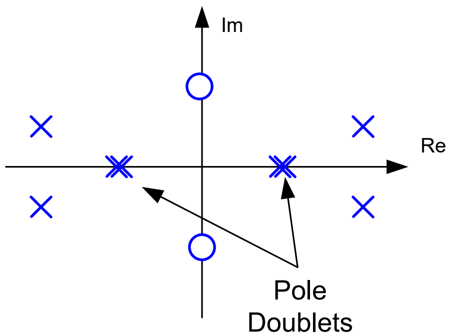
Review from Last Time

$$H_A(\omega^2) = \frac{H_0^2\left[(\omega-z_1)(\omega-z_2)\bullet\ldots\bullet(\omega-z_m)\right]\bullet\left[(\omega+z_1)(\omega+z_2)\bullet\ldots\bullet(\omega+z_m)\right]}{\left[(\omega-p_1)(\omega-p_2)\bullet\ldots\bullet(\omega-p_n)\right]\bullet\left[(\omega+p_1)(\omega+p_2)\bullet\ldots\bullet(\omega+p_n)\right]}$$

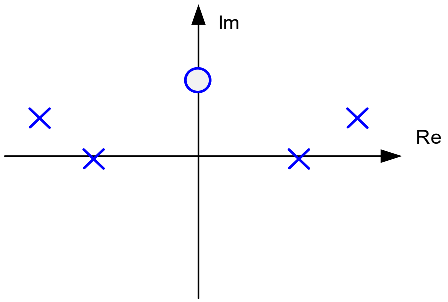
If inverse exists

$$T_{AM}(s) = \frac{H_0(s-jz_1)(s-jz_2)\bullet\ldots\bullet(s-jz_m)}{(s-jp_1)(s-jp_2)\bullet\ldots\bullet(s-jp_n)}$$

Example:

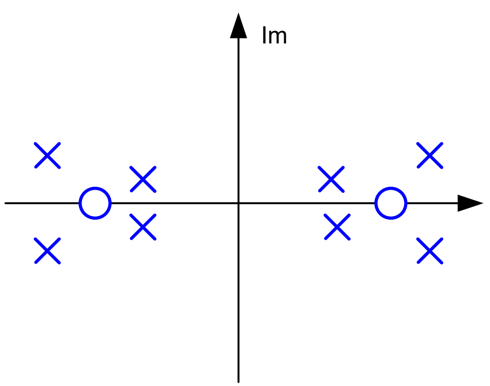


Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$

Example:



Inverse does not exist because zeros are of odd multiplicity on the real axis

$$H_A\left(\omega^2\right)=\frac{H_0^2\left[(\omega\text{-}z_1)(\omega\text{-}z_2)\bullet\ldots\bullet(\omega\text{-}z_m)\right]\bullet\left[(\omega+z_1)(\omega+z_2)\bullet\ldots\bullet(\omega+z_m)\right]}{\left[(\omega\text{-}p_1)(\omega\text{-}p_2)\bullet\ldots\bullet(\omega\text{-}p_n)\right]\bullet\left[(\omega+p_1)(\omega+p_2)\bullet\ldots\bullet(\omega+p_n)\right]}$$

If inverse exists

$$T_{AM}(s)=\frac{H_0(s\text{-}jz_1)(s\text{-}jz_2)\bullet\ldots\bullet(s\text{-}jz_m)}{(s\text{-}jp_1)(s\text{-}jp_2)\bullet\ldots\bullet(s\text{-}jp_n)}$$

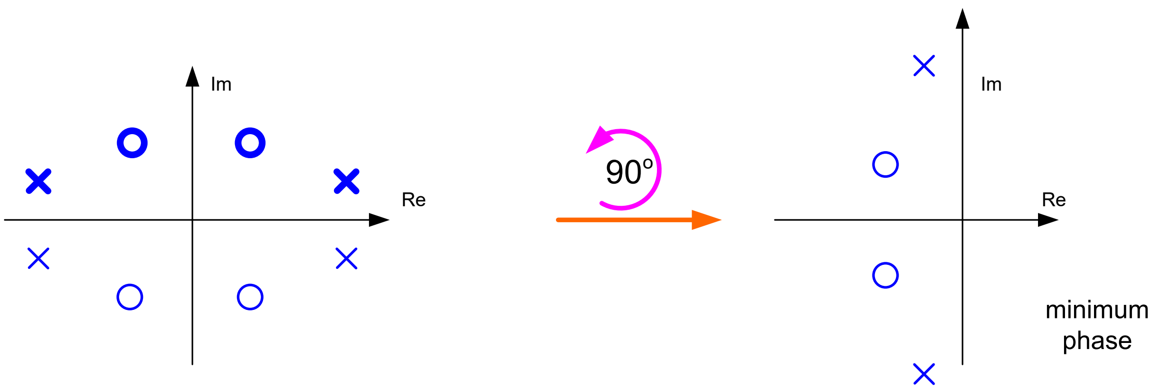
Observations:

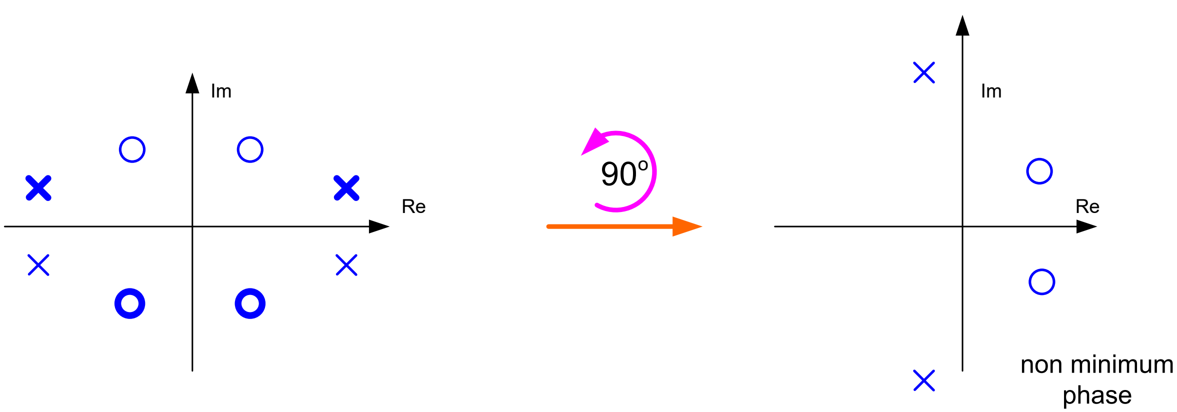
- Coefficients of $T_{AM}(s)$ are real
- If x is a root of $H_A(\omega^2)$, then jx is a root of $T_{AM}(s)$
- Multiplying a root by j is equivalent to rotating it by 90^o cc in the complex plane
- Roots of $T_{AM}(s)$ are obtained from roots of $H_A(\omega^2)$ by multiplying by j
- Roots of $T_{AM}(s)$ are upper half-plane roots and exactly half of real axis roots all rotated cc by 90^o
- If a root of $H_A(\omega^2)$ has odd multiplicity on the real axis, the inverse mapping does not exist
- Other (often many) inverse mappings exist but are not minimum phase
  (These can be obtained by reflecting any subset of the zeros or poles around the imaginary axis into the RHP)

$$H_A\left(\omega^2\right)=\frac{H_0^2\left[(\omega-z_1)(\omega-z_2)\bullet\ldots\bullet(\omega-z_m)\right]\bullet\left[(\omega+z_1)(\omega+z_2)\bullet\ldots\bullet(\omega+z_m)\right]}{\left[(\omega-p_1)(\omega-p_2)\bullet\ldots\bullet(\omega-p_n)\right]\bullet\left[(\omega+p_1)(\omega+p_2)\bullet\ldots\bullet(\omega+p_n)\right]}$$

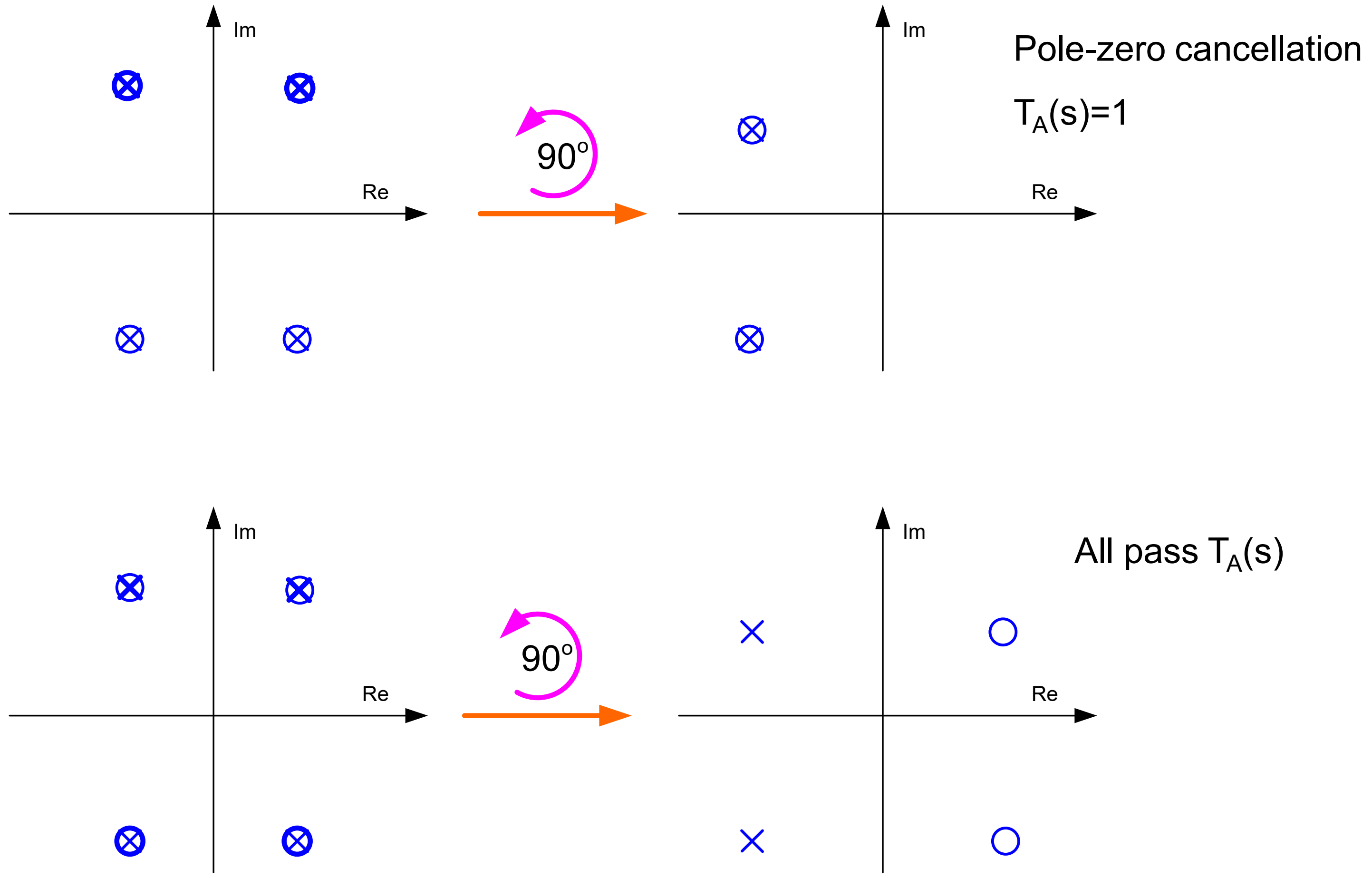
If inverse exists

$$T_{AM}(s)=\frac{H_0(s-jz_1)(s-jz_2)\bullet\ldots\bullet(s-jz_m)}{(s-jp_1)(s-jp_2)\bullet\ldots\bullet(s-jp_n)}$$





All pass functions (and factors)



- Must not allow cancellations to take place in $H_A(\omega^2)$ to obtain all-pass $T_A(s)$
- All-pass $T_A(s)$ is not minimum phase

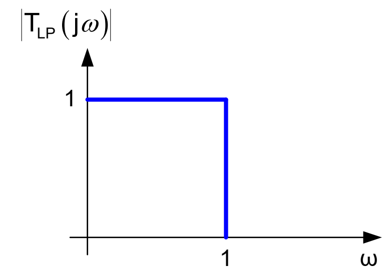
Magnitude Squared Approximating Functions

How is a magnitude-squared approximating function obtained?

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{2l} c_i \omega^{2i}}{\sum_{i=0}^{2k} d_i \omega^{2i}}$$

- Analytical formulations
- Computer-aided optimization

The Approximation Problem



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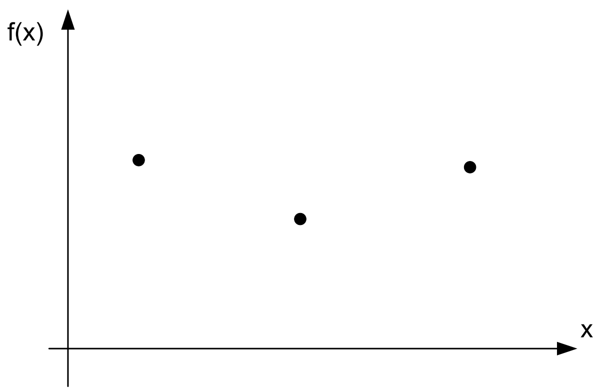
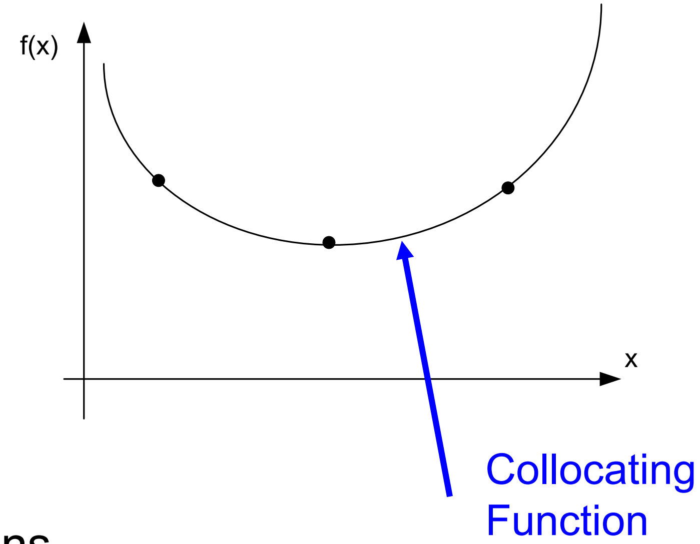
  - $\rightarrow$ Bessel
  - $\rightarrow$ Thompson

Collocation

Collocation is the fitting of a function to a set of points (or measurements) so that the function agrees with the sample at each point in the set.





Often consider critically constrained functions

The function that is of interest for using collocation when addressing the approximation problem is $H_A(\omega^2)$

Collocation

Example: Collocation points $\{(x_1,y_1), (x_2,y_2),(x_3,y_3)\}$

Polynomial collocating function (critically constrained)

$$f(x) = a_0 + a_1 x + a_2 x^2$$

Unknowns: $\{a_1,a_2,a_3\}$

Set of equations:

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2$$
$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2$$
$$y_3 = a_0 + a_1 x_3 + a_2 x_3^2$$

These equations are linear in the unknowns $\{a_1,a_2,a_3\}$

Can be expressed in matrix form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{X} \bullet \mathbf{A}$$

Solution:

$$\mathbf{A} = \mathbf{X}^{-1} \bullet \mathbf{Y}$$

Closed form solution exists when collocating to a polynomial

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1,y_1),(x_2,y_2)...(x_k,y_k)\}$$

$$f(x)=\frac{a_0+a_1x+a_2x^2+...+a_mx^m}{1+b_1x+b_2x^2+...+b_nx^n}$$

where k=m+n+1

The rational fraction is nonlinear in x !

$$y_1\left(1+b_1x_1+b_2x_1^2+...+b_nx_1^n\right)=a_0+a_1x_1+a_2x_1^2+...+a_mx_1^n$$

This can be expressed as

$$y_1=a_0+a_1x_1+a_2x_1^2+...+a_mx_1^n-b_1x_1y_1-b_2x_1^2y_1-...-b_nx_1^ny_1$$

Note this equation is linear in the unknowns $\{a_0,a_1,\ldots a_m,b_1,b_2,\ldots b_n\}$

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \ldots (x_k, y_k)\}$$

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n}$$

where k=m+n+1

$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_m x_1^m - b_1 x_1 y_1 - b_2 x_1^2 y_1 - \ldots - b_n x_1^n y_1$$

$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_m x_2^m - b_1 x_2 y_2 - b_2 x_2^2 y_2 - \ldots - b_n x_2^n y_2$$

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$$y_k = a_0 + a_1 x_k + a_2 x_k^2 + \ldots + a_m x_k^m - b_1 x_k y_k - b_2 x_k^2 y_k - \ldots - b_n x_k^n y_k$$

Collocation

Is it possible to get a closed-form solution when collocating to a rational fraction?

$$\{(x_1, y_1), (x_2, y_2) \ldots (x_k, y_k)\}$$

$$f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n}$$

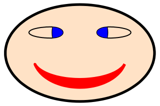
$$y_1 = a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_m x_1^m - b_1 x_1 y_1 - b_2 x_1^2 y_1 - \ldots - b_n x_1^n y_1$$
$$y_2 = a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_m x_2^m - b_1 x_2 y_2 - b_2 x_2^2 y_2 - \ldots - b_n x_2^n y_2$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$y_k = a_0 + a_1 x_k + a_2 x_k^2 + \ldots + a_m x_k^m - b_1 x_k y_k - b_2 x_k^2 y_k - \ldots - b_n x_k^n y_k$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 \; x_1 \; x_1^2 \ldots x_1^m - x_1 y_1 - x_1^2 y_1 - \ldots - x_1^n y_1 \\ 1 \; x_2 \; x_2^2 \ldots x_2^m - x_2 y_2 - x_2^2 y_2 - \ldots - x_2^n y_2 \\ \bullet \\ \bullet \\ 1 \; x_k \; x_k^2 \ldots x_k^m - x_k y_k - x_k^2 y_k - \ldots - x_k^n y_k \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ .. \\ a_m \\ b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{Z} \bullet \mathbf{C}$$

$$\mathbf{C} = \mathbf{Z}^{-1} \bullet \mathbf{Y}$$

Closed form solution when collocating to a rational fraction !



Collocation

Applying to $H_A\left(\omega^2\right)$

$$\left\{(\omega_1, y_1), (\omega_2, y_2) \ldots (\omega_k, y_k)\right\}$$

$$H_A\left(\omega^2\right) = \frac{a_0 + a_1\omega^2 + a_2\omega^4 + \ldots + a_m\omega^{2m}}{1 + b_1\omega^2 + b_2\omega^4 + \ldots + b_n\omega^{2n}}$$

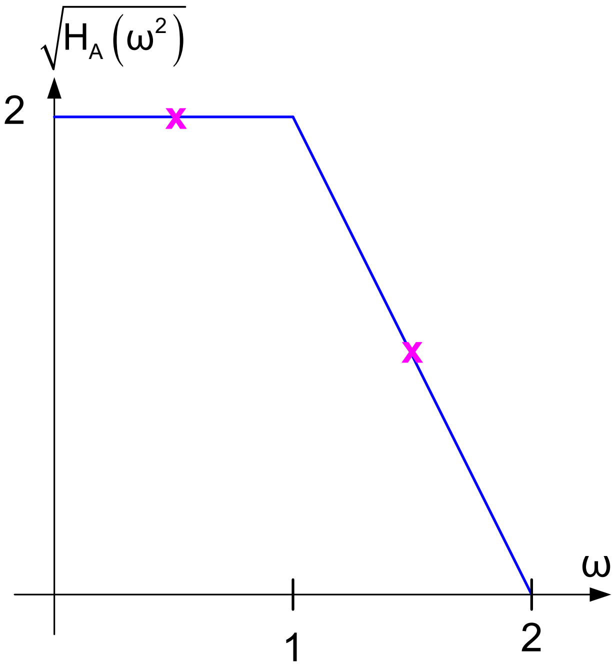
$$\begin{bmatrix} y_1 \\ y_2 \\ \bullet \\ \bullet \\ y_k \end{bmatrix} = \begin{bmatrix} 1 & \omega_1^2 & \omega_1^4 \ldots \omega_1^{2m} & -\omega_1^2 y_1 & -\omega_1^4 y_1 & -\ldots & -\omega_1^{2n} y_1 \\ 1 & \omega_2^2 & \omega_2^4 \ldots \omega_2^{2m} & -\omega_2^2 y_1 & -\omega_2^4 y_1 & -\ldots & -\omega_2^{2n} y_1 \\ \bullet \\ \bullet \\ 1 & \omega_k^2 & \omega_k^4 \ldots \omega_k^{2m} & -\omega_k^2 y_1 & -\omega_k^4 y_1 & -\ldots & -\omega_k^{2n} y_1 \end{bmatrix} \bullet \begin{bmatrix} a_0 \\ a_1 \\ .. \\ a_m \\ b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{Z} \bullet \mathbf{C}$$

$$\mathbf{C} = \mathbf{Z}^{\mathbf{-1}} \bullet \mathbf{Y}$$

Collocation

Example:



x denotes collocation points

$$H_A(\omega^2) = \frac{a_0}{1+b_1\omega^2}$$

$$\left.\begin{aligned} 4 &= \frac{a_0}{1+b_1\left(\tfrac{1}{2}\right)^2} \\ 1 &= \frac{a_0}{1+b_1\left(\tfrac{3}{2}\right)^2} \end{aligned}\right\} \Rightarrow H_A(\omega^2) = \frac{32/5}{1+\left(12/5\right)\omega^2}$$

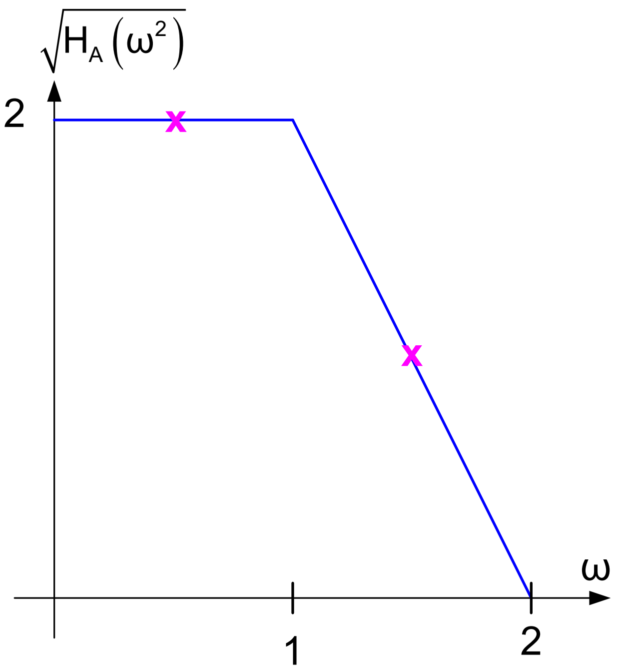
poles at $\quad s = \pm j\sqrt{5/12}$

Collocation

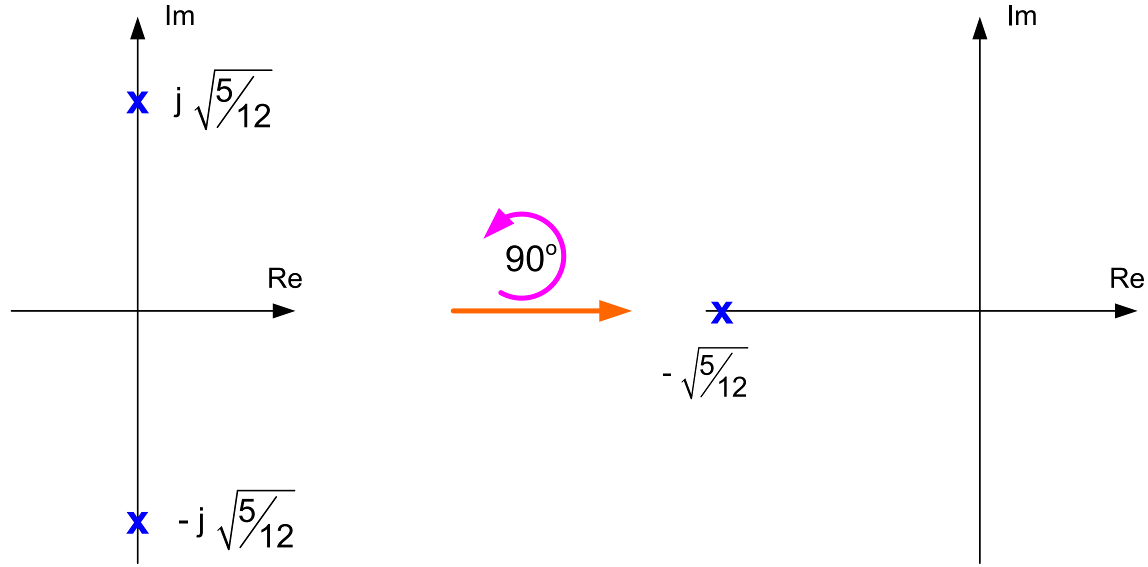
Example:

poles at

$$s = \pm j\sqrt{5/12}$$



x denotes collocation points



Roots of $H_A(\omega^2)$

Roots of $T_{AM}(s)$

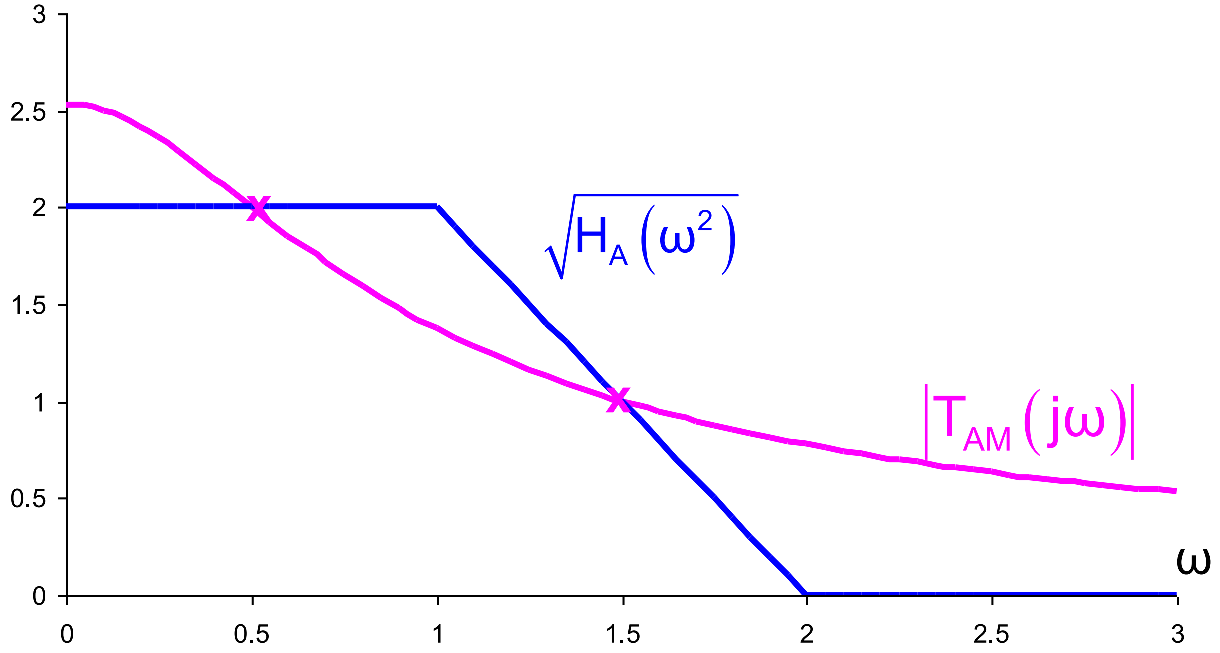
$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$

Collocation

Example:

$$H_A(\omega^2) = \frac{32/5}{1+(12/5)\omega^2}$$

$$T_{AM}(s) = \frac{\sqrt{8/3}}{s+\sqrt{5/12}}$$



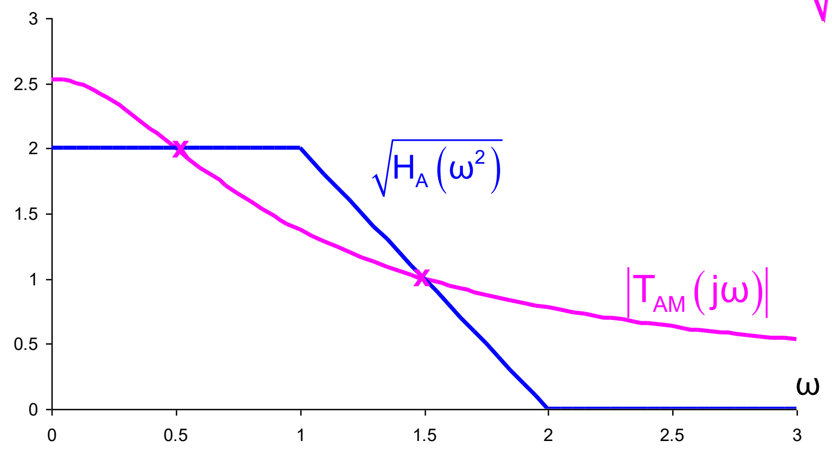
The approximation is reasonable but not too good

Collocation

Example:

$$H_A\left(\omega^2\right) = \frac{32/5}{1+\left(12/5\right)\omega^2}$$

$$T_{AM}(s) = \frac{\sqrt{8/3}}{s + \sqrt{5/12}}$$



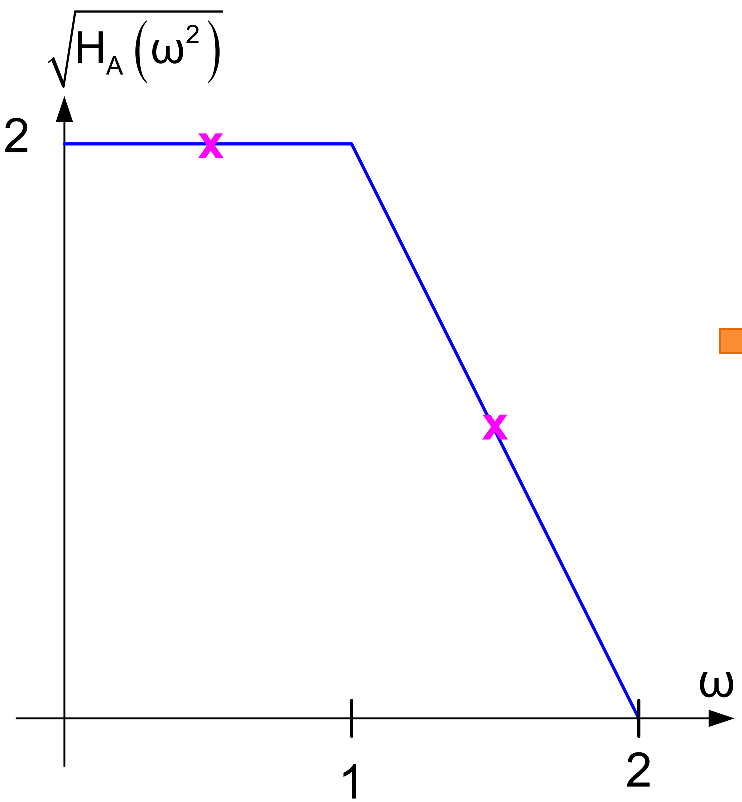
- The problem was critically constrained from a function viewpoint (two variables and two equations)

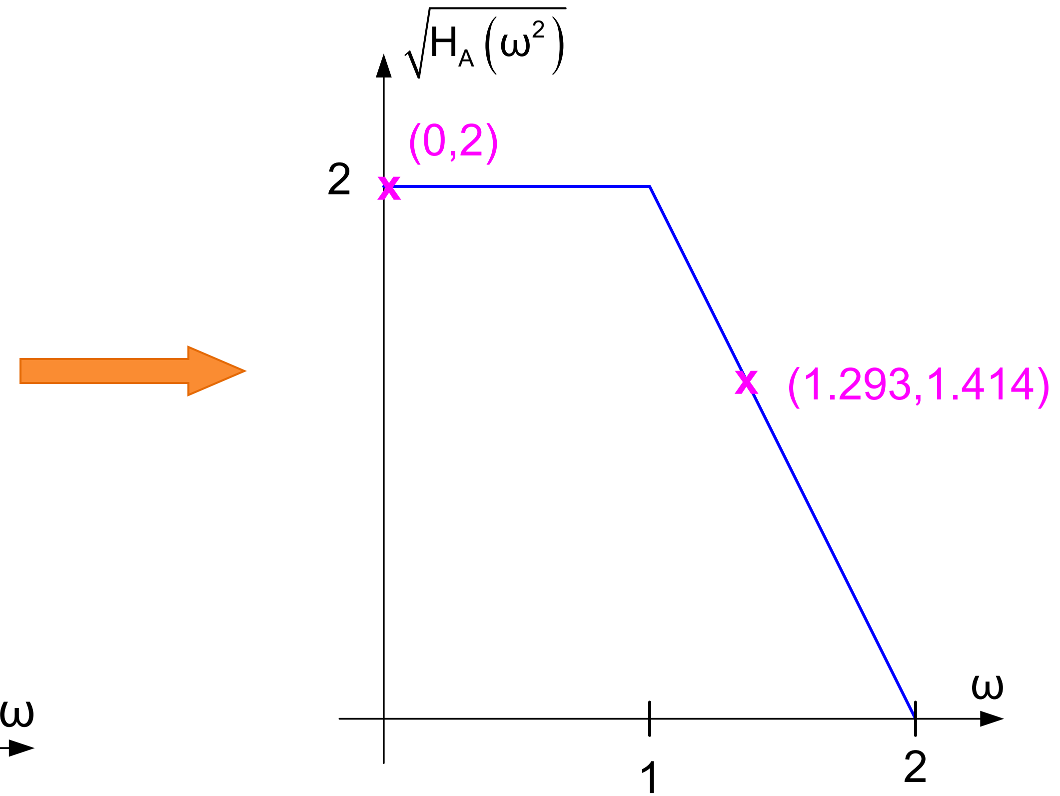
- Highly under-constrained as an approximation technique since the collocation points are also variables

Collocation

Example: same $H_A\left(\omega^2\right)$ but with different collocation points

$$H_A\left(\omega^2\right)=\frac{a_0}{1+b_1\omega^2}$$

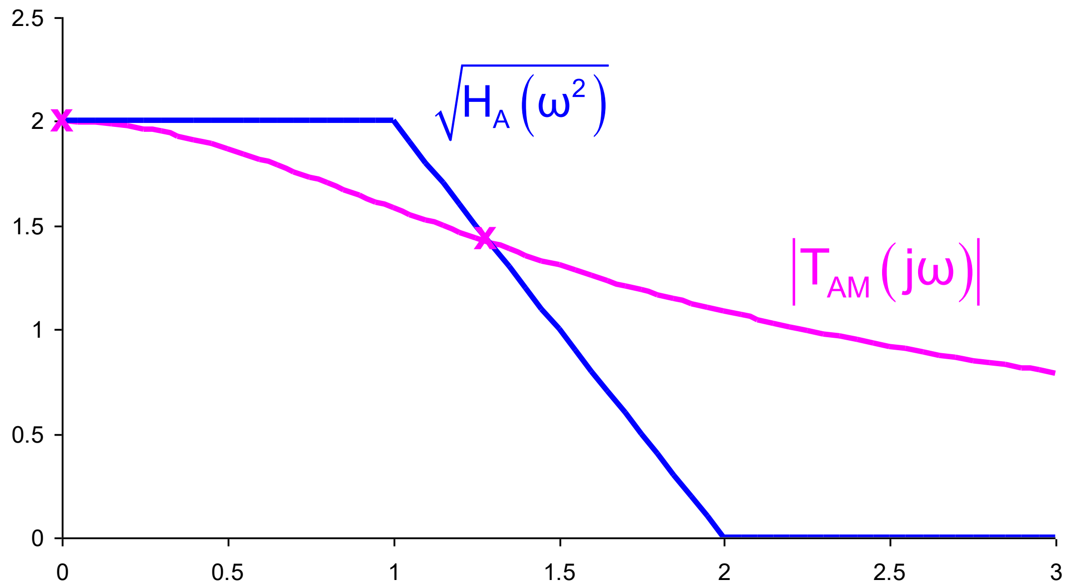




Collocation

Example: same $H_A(\omega^2)$ but with different collocation points $\quad H_A(\omega^2) = \dfrac{a_0}{1+b_1\omega^2}$

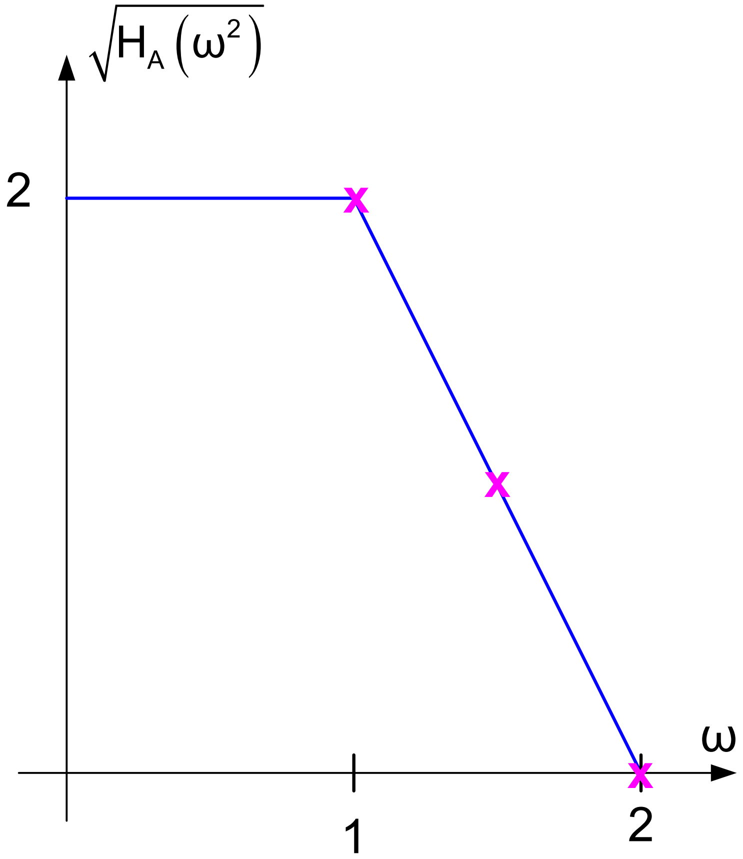
$$\left.\begin{aligned} 4 &= \frac{a_0}{1+b_1(0)^2} \\ 2 &= \frac{a_0}{1+b_1(1.293)^2} \end{aligned}\right\} \Rightarrow \quad H_A(\omega^2) = \frac{4}{1+(.598)\omega^2} \qquad T_{AM}(s) = \frac{1.293}{s+1.293}$$



Choice of collocation points plays a big role on the approximation

Collocation

Example: same $H_A\left(\omega^2\right)$ but with different collocation points and different approximating function



$$H_A\left(\omega^2\right) = \frac{a_0 + a_1\omega^2}{1+b_1\omega^2}$$

$$\left.\begin{aligned} 4 &= \frac{a_0 + a_1}{1+b_1} \\ 1 &= \frac{a_0 + a_1(3/2)^2}{1+b_1(3/2)^2} \\ 0 &= \frac{a_0 + a_1(4)}{1+b_1(4)} \end{aligned}\right\} \Rightarrow \quad H_A\left(\omega^2\right) = \frac{-80 + 20\omega^2}{1+ -16\omega^2}$$

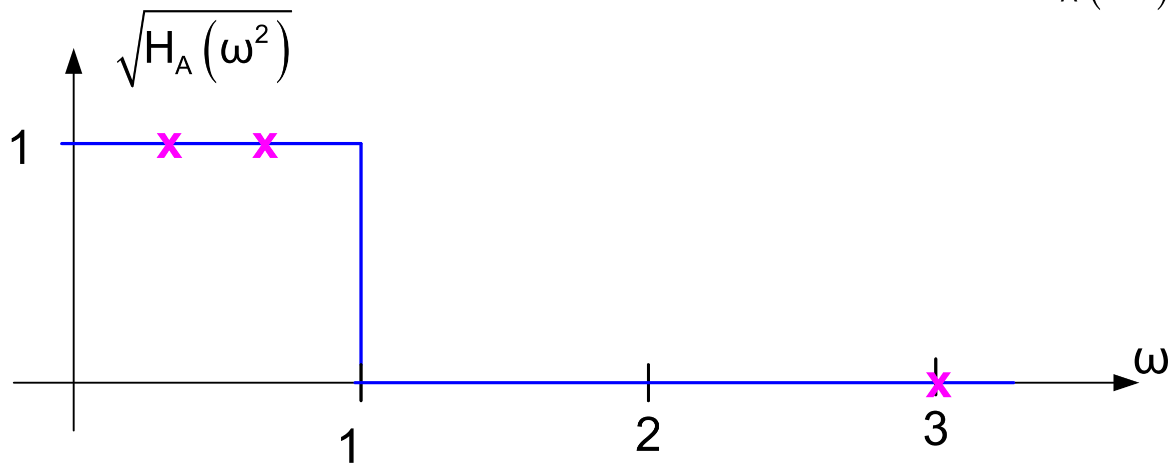
a_0=-80, a_1=20, b_1=-16

Inverse mapping does not exist because roots of odd multiplicity on real axis

Collocation

Example:

$$H_A\left(\omega^2\right) = \frac{a_0 + a_1\omega^2}{1+b_1\omega^2}$$



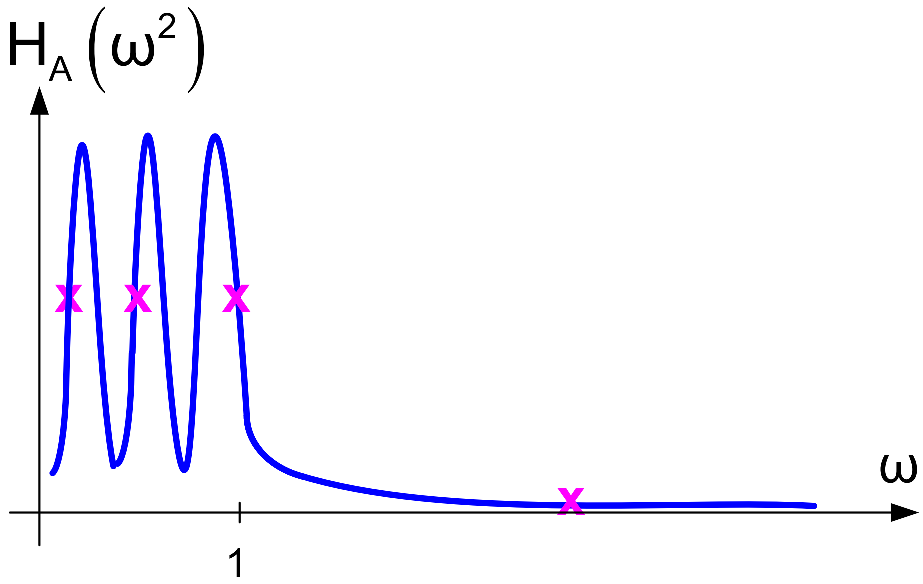
$$\left.\begin{aligned} 1 &= \frac{a_0 + a_1(1/9)}{1+b_1(1/9)} \\ 1 &= \frac{a_0 + a_1(4/9)}{1+b_1(4/9)} \\ 0 &= \frac{a_0 + a_1(9)}{1+b_1(9)} \end{aligned}\right\} \Rightarrow \quad H_A\left(\omega^2\right) = \frac{1+(-27/243)\omega^2}{1+(-27/243)\omega^2}$$

$a_0=1$, $a_1=-27/243$, $b_1=-27/243$

- This solution is equal to 1 at all frequencies except ω=3 where it is undefined
- Thus there is no solution with these collocation points

Collocation

Example:



In some situations, collocation causes a lot of ripple between the collocation points

Collocation Observations

Fitting an approximating function to a set of data or points (collocation points)

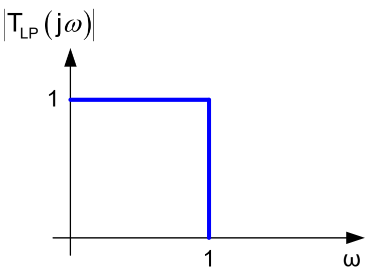
- Closed-form matrix solution for fitting to a rational fraction in ω^2
- Can be useful when somewhat nonstandard approximations are required
- Quite sensitive to collocation points
- Although function is critically constrained, since collocation points are variables, highly under constrained as an optimization approach
- Although fit will be perfect at collocation points, significant deviation can occur close to collocation points
- Inverse mapping to $T_A(s)$ may not exist
- Solution may not exist at specified collocation points

Collocation

What is the major contributor to the limitations observed with the collocation approach?

- Totally dependent upon the value of the desired response at a small but finite set of points (no consideration for anything else)
- Highly dependent upon value of approximating function at a single point or at a small number of points
- Highly dependent upon the collocation points

The Approximation Problem



Approach we will follow:

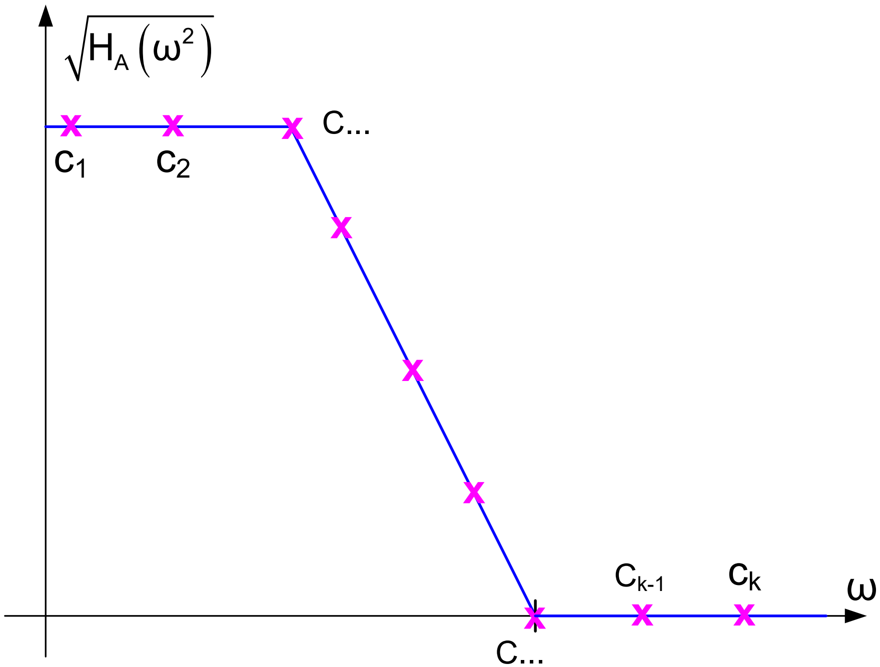
- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares (Cost Function Minimizations)
- Pade Approximations
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Bessel
  - $\rightarrow$ Thompson

Cost Function Minimizations

To minimize the heavy dependence on a small number of points, will consider many points thus creating an over-constrained system

$$H_A(\omega^2) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1 + \sum_{i=1}^{n} b_i \omega^{2i}}$$

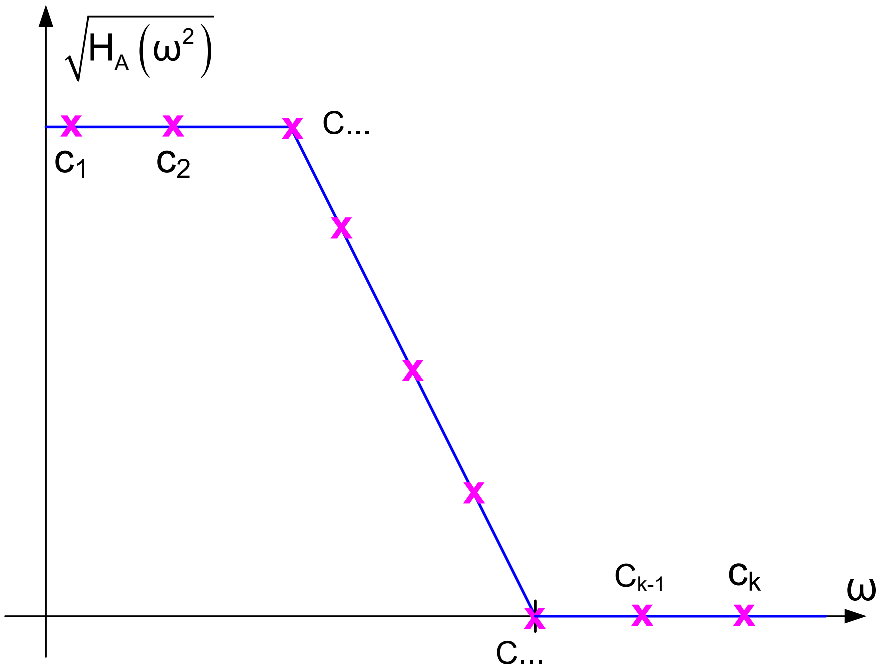
$$k > m+n+1$$



Approximating function can not be forced to go through all points
But, it can be "close" to all points in some sense

Cost Function Minimizations

$$H_A(\omega^2) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1 + \sum_{i=1}^{n} b_i \omega^{2i}}$$



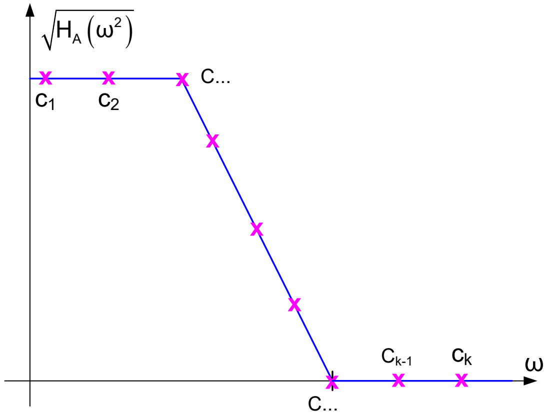
Define the error at point i by

$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$

where $H_D(\omega_i)$ is the desired magnitude squared response at ω_i and where $H_A(\omega_i)$ is the magnitude squared response of the approximating function

Cost Function Minimizations

$$H_A(\omega^2) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1+\sum_{i=1}^{n} b_i \omega^{2i}}$$



$$\varepsilon_i = H_D(\omega_i) - H_A(\omega_i)$$

Goal is to minimize some metrics associated with ε_i at a large number of points

Some possible cost functions

$$C_1 = \sum_{i=1}^{N} |\varepsilon_i| \qquad C_2 = \sum_{i=1}^{N} \varepsilon_i^2$$

$$C_3 = \sum_{i=1}^{N} w_i \varepsilon_i^2 \qquad C_{w:m} = \sum_{i=1}^{N} w_i |\varepsilon_i|^m$$

$$C_{w:m_1,m_2} = \sum_{i=1}^{N_1} w_i |\varepsilon_i|^{m_1} + \sum_{i=N_1+1}^{N} w_i |\varepsilon_i|^{m_2}$$

w_i a weighting function

Termed "L_m norm" if exponent is m and weight is 1

- Reduces emphasis on individual points
- Some much better than others from performance viewpoint
- Some much better than others from computation viewpoint
- **Realization of no concern how approximation obtained, only of how good it is !**

Least Squares Approximation

$$H_A\left(\omega^2\right) = \frac{\sum_{i=0}^{m} a_i \omega^{2i}}{1+\sum_{i=1}^{n} b_i \omega^{2i}} \qquad \varepsilon_i = H_D\left(\omega_i\right) - H_A\left(\omega_i\right)$$

Consider:

$$C_3 = \sum_{i=1}^{N} w_i \varepsilon_i^2$$

w_i a weighting function

If exponent in cost function is 2, termed "least squares" cost function

Least Mean Square (LMS) based cost functions have minimums that can be analytically determined for some useful classes of approximating functions $H_A(\omega^2)$

- Often termed a L_2 norm
- Minimizing L_1 norm often provides better approximation but no closed-form analytical expressions
- Most of the other metrics listed on previous slide are not easy to get closed-form expressions for minimums though computer optimization can be used: may be plagued by multiple local minimums but they may still be useful

Regression Analysis Review

Consider an nth order polynomial in x

$$F(x) = \sum_{k=0}^{n} a_k x^k$$

Consider N samples of a function $\tilde{F}(x)$

$$\hat{F}(x) = \left\langle \tilde{F}(x_i) \right\rangle_{i=1}^{N}$$

where the sampling coordinate variables are

$$X = \langle x_i \rangle_{i=1}^{N}$$

Define the summed square difference cost function as

$$C = \sum_{i=0}^{N} \left(F(x_i) - \tilde{F}(x_i) \right)^2$$

A standard regression analysis can be used to minimize C with respect to $\{a_0, a_1, \ldots a_n\}$

To do this, take the n+1 partials of C wrt the a_i variables

Regression Analysis Review

$$C = \sum_{i=0}^{N} \left(F(x_i) - \tilde{F}(x_i) \right)^2 \qquad F(x) = \sum_{k=0}^{n} a_k x^k$$

$$C = \sum_{i=0}^{N} \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right)^2$$

Taking the partial of C wrt each coefficient and setting to 0, we obtain the set of equations

$$\frac{\partial C}{\partial a_0} = 2\sum_{i=0}^{N} \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right) = 0$$

$$\frac{\partial C}{\partial a_1} = 2\sum_{i=0}^{N} x_i^1 \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right) = 0$$

$$\frac{\partial C}{\partial a_2} = 2\sum_{i=0}^{N} x_i^2 \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right) = 0$$

$$\cdots$$

$$\frac{\partial C}{\partial a_n} = 2\sum_{i=0}^{N} x_i^n \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right) = 0$$

This is linear in the a_ks.

$$X \bullet A = F$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \ldots \\ a_n \end{bmatrix}$$

Solution is

$$A = X^{-1} \bullet F$$

Regression Analysis Review

A few details about regression analysis:

$$X \bullet A = F$$

$$A = X^{-1} \bullet F$$

$$X = \begin{bmatrix} N+1 & \sum_{i=0}^{N} X_i & \sum_{i=0}^{N} X_i^2 & \cdots & \sum_{i=0}^{N} X_i^n \\ \sum_{i=0}^{N} X_i & \sum_{i=0}^{N} X_i^2 & \cdots & & \sum_{i=0}^{N} X_i^{n+1} \\ \sum_{i=0}^{N} X_i^2 & \cdots\cdot & & & \sum_{i=0}^{N} X_i^{n+2} \\ \cdots & & & & \\ \sum_{i=0}^{N} X_i^n & \sum_{i=0}^{N} X_i^{n+1} & \cdots & & \sum_{i=0}^{N} X_i^{2n} \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_n \end{bmatrix}$$

$$F = \begin{bmatrix} \sum_{i=0}^{N} \tilde{F}(x_i) \\ \sum_{i=0}^{N} x_i \tilde{F}(x_i) \\ \sum_{i=0}^{N} x_i^2 \tilde{F}(x_i) \\ \cdots \\ \sum_{i=0}^{N} x_i^n \tilde{F}(x_i) \end{bmatrix}$$

Regression Analysis Review

$$C = \sum_{i=0}^{N} \left(F(x_i) - \tilde{F}(x_i) \right)^2 \qquad F(x) = \sum_{k=0}^{n} a_k x^k$$

$$C = \sum_{i=0}^{N} \left(\sum_{k=0}^{n} a_k x_i^k - \tilde{F}(x_i) \right)^2$$

$$A = X^{-1} \bullet F$$

Observations about Regression Analysis:

- Closed form solution
- Requires inversion of a (n+1) dimensional square matrix
- Not highly sensitive to any single measurement
- Widely used for fitting a set of data to a polynomial model
- Points need not be uniformly distributed
- Adding weights does not complicate solution

This analysis was restricted to a polynomial – will see how applicable it is to a rational fraction !

Stay Safe and Stay Healthy !

End of Lecture 7